صيغة ليجندر التضاعفية:

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}} \cdot \Gamma(s) \cdot \Gamma\left(s + \frac{1}{2}\right)$$

حيث $s > 0$.

الحل: لدينا $\beta(p, q) = \frac{\Gamma(p) \cdot \Gamma(q)}{\Gamma(p+q)}$

لنفرض أن $p = q = s$

$$\Rightarrow \frac{(\Gamma(s))^2}{\Gamma(2s)} = \beta(s, s) = 2 \int_0^{\pi/2} (\sin\varphi)^{2s-1} \cdot (\cos\varphi)^{2s-1} \, d\varphi \Rightarrow$$

الشكل المثلثي

$$\boxed{\beta(p, q) = 2 \int_0^{\pi/2} (\sin\varphi)^{2p-1} \cdot (\cos\varphi)^{2q-1} \, dx}$$

$$\Rightarrow \quad = 2 \int_0^{\pi/2} \left(\frac{\sin 2\varphi}{2} \right)^{2s-1} d\varphi$$

$$= \frac{2}{2^{2s-1}} \int_0^{\pi/2} (\sin 2\varphi)^{2s-1} \, d\varphi$$

نفرض $2\varphi = t \Leftarrow \varphi = \frac{t}{2} \Leftarrow d\varphi = \frac{dt}{2}$

وبالتالي:

$$\frac{[\Gamma(s)]^2}{\Gamma(2s)} = \frac{2}{2^{2s-1}} \int_0^{\pi} (\sin t)^{2s-1} \frac{dt}{2}$$

$$= \frac{1}{2^{2s-1}} \int_0^{\pi} (\sin t)^{2s-1} \, dt \Rightarrow \int_0^{\frac{\pi}{2}} \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{2}{2^{2s-1}} \int_0^{\frac{\pi}{2}} (\sin t)^{2s-1} \, dt$$

تناظر مع شكل هندسي لـ β

$$= \frac{1}{2^{2s-1}} \beta\left(s, \frac{1}{2}\right) = \frac{1}{2^{2s-1}} \cdot \frac{\Gamma(s) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(s + \frac{1}{2}\right)}$$

الموضوع :

$$\Rightarrow \frac{[\Gamma(S)]^2}{\Gamma(2S)} = \frac{\sqrt{\pi}}{2^{2S-1}} \cdot \frac{\Gamma(S)}{\Gamma(S+\frac{1}{2})}$$

$$\Rightarrow \frac{\Gamma(S)}{\Gamma(2S)} = \frac{\sqrt{\pi}}{2^{2S-1}} \cdot \frac{1}{\Gamma(S+\frac{1}{2})}$$

$$\Rightarrow \Gamma(2S) = \frac{2^{2S-1}}{\sqrt{\pi}} \cdot \Gamma(S) \cdot \Gamma(S+\tfrac{1}{2})$$

تمارين :

-1- أثبت أنّ : $\Gamma(n+\frac{1}{2}) = \frac{(2n)!\sqrt{\pi}}{n! \cdot 4^n}$ حيث $n \in \mathbb{N}$

الحل : باستخدام صيغة بجندر من أجل $S=n$ نجد :

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}} \Gamma(n) \cdot \Gamma(n+\tfrac{1}{2})$$

$$(2n-1)! = \frac{2^{2n-1}}{\sqrt{\pi}} (n-1)! \cdot \Gamma(n+\tfrac{1}{2})$$

$$\Rightarrow \Gamma(n+\tfrac{1}{2}) = \frac{(2n-1)! \cdot \sqrt{\pi}}{(n-1)! \cdot 2^{n-1}}$$

نضرب البسط و المقام بـ $2n$

$$= \frac{(2n)! \cdot \sqrt{\pi}}{n! \cdot 2^{2n}} = \frac{(2n)! \cdot \sqrt{\pi}}{n! \cdot 4^n}$$

-2- أثبت أنّ $I = \int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi}$

الحل :

$I = 2\int_0^{+\infty} e^{-t^2} dt$ نتناظر $\Gamma(S) = 2\int_0^{+\infty} e^{2S-1} \cdot e^{-t^2} dt$

$= \Gamma(\frac{1}{2}) = \sqrt{\pi}$ $\quad 2S-1=0$

$$\boxed{S = \frac{1}{2}}$$

الموضوع:

-3- احسب قيمة التكامل

$$I = \int_0^{\pi/2} (tg\,x)^n \, dx$$

; $0 < n < 1$

الحل:

$$= \int_0^{\pi/2} (\sin x)^n \cdot (\cos x)^{-n} \, dx$$

بالمقارنة مع الشكل المثلثي للتابع β :

$$\beta(p, q) = 2 \int_0^{\pi/2} (\sin\varphi)^{2p-1} \cdot (\cos\varphi)^{2q-1} \, dx$$

وبالتالي

$$p = \frac{n+1}{2} \Leftarrow 2p - 1 = n$$

$$q = \frac{-n+1}{2} \qquad 2q - 1 = -n$$

$$I = \frac{1}{2} \beta\left(\frac{n+1}{2}, \frac{-n+1}{2} \right).$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sin\left(\left(\frac{n+1}{2}\right)\pi\right)}$$

$$= \frac{\pi}{2\cos\left(\frac{n}{2}\pi\right)}$$

-4- احسب قيمة التكامل

$$I = \int_0^{+\infty} \frac{\sqrt{t}}{(1+t^2)^3} \, dt$$

الحل: نفرض $t^2 = y$ أو $t = \sqrt{y} \Rightarrow dt = \frac{1}{2} y^{-\frac{1}{2}} dy$

$$\Rightarrow I = \frac{1}{2} \int_0^{+\infty} \frac{y^{\frac{1}{4}}}{(1+y)^3} \, y^{-\frac{1}{2}} \, dy$$

$$I = \frac{1}{2} \int_0^{+\infty} \frac{y^{-\frac{1}{4}}}{(1+y)^3} \, dy$$

الموضوع: ....

بالمقارنة مع الشكل الكسري للتابع β:

$$\beta(p,q) = \int_0^{+\infty} \frac{y^{p-1}}{(1+y)^{p+q}}\, dy$$

وبالتالي: $p-1 = -\frac{1}{4} \Leftarrow \quad p = \frac{3}{4} \quad \& \quad q = \frac{9}{4}$

$p+q = 3$

$$\Rightarrow I = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{9}{4}\right) = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{3}{4}\right) \cdot \Gamma\left(\frac{9}{4}\right)}{\Gamma(3)}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right) \cdot \frac{5}{4} \Gamma\left(\frac{5}{4}\right)}{2!}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right) \cdot \frac{5}{4} \cdot \frac{1}{4} \Gamma\left(\frac{1}{4}\right)}{2} \qquad \Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin}$$

$$= \frac{5}{64} \Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{3}{4}\right)$$

$$= \frac{5}{64} \cdot \frac{\pi}{\underbrace{\sin\frac{\pi}{4}}_{=\sqrt{2}}} = \frac{5\pi\sqrt{2}}{64}$$

تمرين 5: احسب قيمة التكامل:

$$I = \int_0^a x^2 \cdot \sqrt{a^2 - x^2}\, dx \quad ; a > 0$$

الحل:

$$I = a \int_0^a x^2 \sqrt{1 - \frac{x^2}{a^2}}\, dx$$

نفرض $y = \frac{x^2}{a^2} \Rightarrow x = a \cdot y^{\frac{1}{2}}$

$dx = \frac{a}{2} \cdot y^{-\frac{1}{2}} dy$

$$I = a \int_0^1 a^2 \cdot y \cdot \sqrt{1 - y^2} \cdot \frac{a}{2} y^{-\frac{1}{2}} dy$$

الموضوع:

$$I = \frac{a^4}{2} \int_0^1 y^{\frac{1}{2}} (1-y)^{\frac{1}{2}} \, dy$$

مقارنة مع الشكل تابع β نجد:

$$p = q = \frac{3}{2} \Leftarrow p-1 = \frac{1}{2}, \quad q-1 = \frac{1}{2}$$

$$I = \frac{a^4}{2} \beta\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$= \frac{a^4}{2} \frac{\Gamma\left(\frac{3}{2}\right) \cdot \Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}$$

$$= \frac{a^4}{4} \left(\frac{1}{2} \Gamma\left(\frac{1}{2}\right)\right)^2$$

$$= \frac{a^4 \pi}{16}$$

6- احسب قيمة التكامل:

$$I = \int_0^2 \frac{x^2}{\sqrt{2-x}} \, dx$$

الجواب $= \frac{64}{15}\sqrt{2}$

7- احسب قيمة التكامل:

$$I = \int_0^{+\infty} 3^{-4x^2} \, dx$$

الحل:

$$3 = e^{\ln 3}$$

$$I = \int_0^{+\infty} \left(e^{\ln 3}\right)^{-4x^2} dx$$

$$= \int_0^{+\infty} e^{-4\ln 3 \cdot x^2} \, dx$$

نفرض $t = 4\ln 3 \, x^2 \Leftarrow x = \frac{t^{\frac{1}{2}}}{2\sqrt{\ln 3}}$

$$dx = \frac{1}{4\sqrt{\ln 3}} \cdot t^{-\frac{1}{2}} \, dt$$

الموضوع:

$$= \frac{1}{4\sqrt{\ln 3}} \int_0^{+\infty} e^{-t} \cdot t^{-\frac{1}{2}} dt = \frac{1}{4\sqrt{\ln 3}} \int_0^{+\infty} t^{-\frac{1}{2}} \cdot e^{-t} \cdot dt$$

بالمقارنة مع التابع Γ نجد التكامل:

$$= \frac{1}{4\sqrt{\ln 3}} \cdot \Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{4\sqrt{\ln 3}}$$

-8- احسب قيمة التكامل

$$I = \int_0^1 (x \cdot \ln x)^3 dx$$

الحل:

نفرض $y = -\ln x \Leftarrow x = e^{-y}$

$$\Rightarrow dx = -e^{-y} dy$$

حدود التكامل: $x \to 0 \Rightarrow y \to +\infty$ ، $x \to 1 \Rightarrow y \to 0$

$$I = -\int_{+\infty}^{0} (-e^{-y} y)^3 e^{-y} dy$$

$$I = +\int_{+\infty}^{0} y^3 \cdot e^{-4y} dy = -\int_0^{+\infty} y^3 \cdot e^{-4y} dy$$

نفرض $4y = t \Rightarrow y = \frac{t}{4} \Rightarrow dy = \frac{dt}{4}$

$$I = -\int_0^{+\infty} \left(\frac{t}{4}\right)^3 e^{-t} \frac{dt}{4}$$

$$= -\frac{1}{256} \int_0^{+\infty} t^3 \cdot e^{-t} dt$$

$$= -\frac{1}{256} \cdot \Gamma(4) = -\frac{3!}{256}$$

$$= \frac{-6}{256} = \frac{-3}{128}$$

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